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THE INVERSE PROBLEM OF STREAMLINING SINGULARITIES WITH THE PLANE FLOW OF AN IDEAL FLUID WITH A FREE BOUNDARY

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UDC 532.59

In the present study we will examine the plane potential steady-state flow of a heavy ideal fluid with a free boundary. In this case, the velocity potential has a finite number of point singularities. The potential motions of the fluid in the presence of a finite system of point singularities were examined in [1] in connection with the problem of the streamlining of a wing-shaped object under water, and the calculation of the wave resistance, and these were dealt with also in [2, 3], etc. In the cited references the problem was solved in "direct" formulation, i.e., on the basis of given point singularities of a complex potential, which simulated the streamlined solid, and the profile of the free surface and the velocity field were determined. The solutions were derived within the framework of linear wave theory. In the present paper we solve the "inverse" problem: the stationary profile of the free surface is given, and we have to reconstruct the pattern of the flow through the thickness of the fluid. The solution of the problem is achieved both in approximate linear theory and in a precise formulation.

1. Construction of the Solution. Let $S(x)$ $(-\infty < x < \infty)$ represent the profile of the free surface and let v_0 be the velocity of the unperturbed flow. By means of $G \subset C$ we will denote the region occupied by the fluid: $G = \{z : z = x + iy,\ y < S(x)\}$ (we will investigate an infinitely deep fluid). The set of points lying at the surface is denoted S: $S = \{z: \operatorname{Im} z = S(\operatorname{Re} z)\}$. We will impose the following limitation on $S(x)$:

$$S(x) < v_0^2/2g \tag{1.1}$$

(g is the gravitational acceleration). Fulfillment of inequality (1.1) ensures the absence of critical points of complex velocity at the boundary of the fluid and this, in turn, guarantees the smoothness of the profile for $S(x)$ [4]. Let P represent the total number of poles for the complex velocity $w(z)$ in G, i.e., $P = \sum_{i:\, z_i \in G} p_i$, where $P < \infty$ and p_i represent the multiplicity of the pole z_i. The quantities P and p_i are not known in advance and are determined in the process of the solution. Moreover, the natural condition of limitation with respect to the velocity of the fluid at infinity is assumed to be satisfied:

$$|w(z)| < w < +\infty,\ |z| > R,\ z \in G \tag{1.2}$$

(R is a rather large number). The boundary conditions for the potential W are satisfied at the free surface:

Lyubertsy. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 30, No. 6, pp. 86-91, November-December, 1989. Original article submitted December 22, 1987; revision submitted June 24, 1988.

$$W = \Phi(z) + i\Psi(z), \tag{1.3}$$

$$\frac{1}{2}\left[\left(\frac{\partial\Phi(x, S(x))}{\partial x}\right)^2 + \left(\frac{\partial\Phi(x, S(x))}{\partial y}\right)^2\right] + \rho g S(x) = c_1, \quad \Psi(x, S(x)) = c_2$$

(ρ is the density of the fluid and c_1, c_2 are constants).

To reconstruct the velocity w(z) in G, we will first of all find its poles $z_k \in G$. For this, in the set $G^+ = C \backslash (G \cup S)$ we will determine the function $R(\lambda)$ by the relationship

$$R(\lambda) = -\int_{-\infty}^{+\infty} \frac{\sqrt{(v_0^2 - 2gS(x))(1 + (S'(x))^2)}\, dx}{(x + iS(x) - \lambda)^2}, \quad S' = \frac{dS}{dx}. \tag{1.4}$$

It is obvious that the function $R(\lambda)$ is holomorphic in G^+. Moreover, it can be demonstrated that the analytical extension R(z) of the function $R(\lambda)$ to the total complex plane C is a rational function [within the assumptions that we have made here with respect to w(z)]. In this case the poles of the function R(z) coincide with those of w(z) in G. Indeed, using the boundary conditions (1.3), and bearing in mind that w(z) = dW/dz, as well as condition (1.1), it is easy to calculate the complex velocity w(z) at the free boundary:

$$w(x, S(x)) = \left[\frac{v_0^2 - 2gS(x)}{1 + (S'(x))^2}\right]^{1/2} (1 - iS'(x)). \tag{1.5}$$

According to (1.5), the expression found in (1.4) for $R(\lambda)$ is written as follows:

$$R(\lambda) = \int_S \frac{w(\xi)\, d\xi}{(\xi - \lambda)^2}, \quad \lambda \in G^+.$$

Based on the familiar theorem from complex analysis [5], with consideration of (1.2) when $\lambda \in G^+$, we obtain the equation

$$R(\lambda) = 2\pi i \sum \operatorname{Res}\left[\frac{w(z)}{(z - \lambda)^2}, z_k\right],$$

where summation is carried out over all poles $z_k \in G$ of the function w(z). Each pole z_k of order p_k yields the following contribution to $R(\lambda)$:

$$\operatorname{Res}\left[\frac{w(z)}{(z - \lambda)^2}, z_k\right] = \sum_{l=2}^{p_k+1} \frac{A_l}{(z_k - \lambda)^l}$$

($A_{p_k+1} \neq 0$). The analytical extension of $R(\lambda)$ from the region G^+ to the total complex plane thus has the form

$$R(z) = 2\pi i \sum_k \frac{p_k(z)}{(z_k - z)^{p_k+1}}$$

[$p_k(z)$ is a polynomial of a degree no higher than $p_k - 1$]. It is obvious that when w(z) exhibits no singular points in G, $R(z) \equiv 0$.

Let us note that within the framework of linear wave theory, for the velocity at the boundary in the place of (1.5) we validly have the expression

$$w(x, 0) = v_0(1 - (g/v_0^2)\, S(x) - iS'(x)). \tag{1.6}$$

Therefore $R(\lambda)$ must be determined by the relationship

$$R(\lambda) = -v_0 \int_{-\infty}^{+\infty} \frac{1 - \nu S(x) - iS'(x)}{(x - \lambda)^2}\, dx, \quad \operatorname{Im}\lambda > 0 \tag{1.7}$$

($\nu = g/v_0^2$). In the following, all considerations proceed analogously.

Thus, the problem of finding the complex-velocity poles w(z) in G reduces to the simpler problem of finding the poles of the rational function R(z), which in turn, within the framework of the standard techniques of the Padé approximations, permits of an exact solution [6]. Let us dwell on this point in somewhat greater detail.

The function $R(\lambda)$ in the vicinity of any point $\lambda_0 \in G^+$ can be expanded into a Taylor series

$$R(\lambda) = \sum_m c_m (\lambda - \lambda_0)^m. \tag{1.8}$$

The convergence radius of series (1.8) is no smaller than the distance from λ_0 to S. The expressions for the coefficients of this expansion are easily obtained in the form of

$$c_m = \int_S \frac{w(\xi)\, d\xi}{(\xi - \lambda_0)^{m+2}}. \tag{1.9}$$

According to the definition in the notation [6] of the Padé approximation [L/M] of the function F(z) is known as the rational function

$$[L/M] = \frac{A^{[L/M]}(z)}{B^{[L/M]}(z)},$$

if there exist polynomials $A^{[L/M]}(z)$ and $B^{[L/M]}(z)$ of degree L and M, respectively, such that

$$[L/M] = F(z) + O(z^{L+M+1}). \tag{1.10}$$

According to (1.10), we will look for the Padé approximation (1.8) in the form

$$\sum c_m z^m = \frac{a_0 + a_1 z + \ldots + a_L z^L}{1 + b_1 z + \ldots + b_M z^M} + O(z^{L+M+1}),$$

where $z = \lambda - \lambda_0$. The polynomial $B^{[L/M]}(z)$, in the denominator of the Padé approximation, with accuracy to the numerical factor, can be represented in the form of the determinant [6]

$$Q^{[L/M]}(z) = \begin{vmatrix} c_{L-M+1} & c_{L-M+2} & \cdots & c_L & c_{L+1} \\ c_{L-M+2} & c_{L-M+3} & \cdots & c_{L+1} & c_{L+2} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ c_L & c_{L+1} & \cdots & c_{L+M-1} & c_{L+M} \\ z^M & z^{M-1} & \cdots & z & 1 \end{vmatrix}. \tag{1.11}$$

The determinant $C(L, M) = Q^{[L/M]}(0)$ vanishes when the order of the approximation exceeds that of the approximate rational function. More precisely, the following theorem is valid.

Let

$$R(z) = \frac{\sum_{i=0}^{l} \alpha_i z^i}{\sum_{j=0}^{m} \beta_j z^j}, \quad \sum_{j=0}^{m} \beta_j \neq 0, \tag{1.12}$$

then

$$C(l+1, m) \neq 0, \; C(l, m+1) \neq 0, \quad C(l+i, m+j) = 0 \text{ when } i, j = 1, 2, \ldots \tag{1.13}$$

Condition (1.13) is also sufficient for R(z) to be a rational function. We can conclude on the basis of this theorem that after calculating the finite number of quantities C(L, M), by means of (1.11) we determine the denominator of the rational function R(z) precisely.

Thus we have found the poles z_k of the function w(z) in region G and their multiple p_k. In order to find w(z) in G, i.e., completely to reconstruct the velocity field in the thickness of the fluid, it is sufficient to use the formula

$$w(z)=\frac{(z-z_0)^N(-1)}{2\pi i\prod_k (z-z_k)^{p_k}}\times$$
$$\times\int_{-\infty}^{+\infty}\frac{\left[\left(v_0^2-2gS(x)\right)\left(1+(S'(x))^2\right)\right]^{1/2}\prod_k (x+iS(x)-z_k)^{p_k}\,dx}{(x+iS(x)-z_0)^N(x+iS(x)-z)} \tag{1.14}$$

in the exact case and to utilize the formula

$$w(z)=\frac{(z-z_0)^N v_0(-1)}{2\pi i\prod_k (z-z_k)^{p_k}}\int_{-\infty}^{+\infty}\frac{(1-\nu S(x)-iS'(x))\prod_k (x-z_k)^{p_k}\,dx}{(x-z_0)^N(x-z)} \tag{1.15}$$

within the scope of the theory of small waves, where $G^+ \ni z_0$ is an arbitrary fixed point, N = P + 1. These formulas are easily obtained by applying the Cauchy theorem to the analytical function within the region

$$f(z)=\frac{w(z)\prod_k (z-z_k)^{p_k}}{(z-z_0)^N},\quad z\in G$$

with consideration of (1.5) and (1.6), respectively.

Let us take note of the fact that in the general case the integrals from (1.9) are taken numerically, for example, provided that S(x) has been specified experimentally. We will illustrate the proposed approach through the simple example in which the solution is achieved in explicit form.

2. Example. The profile of the free surface induced by a point vortex moving at a velocity v_0 at depth h has been found in [1]:

$$S(x)=\frac{\Gamma}{\pi v_0}\int_x^{+\infty}\frac{t\cos\nu(t-x)-h\sin\nu(t-x)}{t^2+h^2}\,dt \tag{2.1}$$

($\nu = g/v_0{}^2$; Γ is the intensity of the vortex). Since this profile has been found within the scope of linear wave theory, it is natural to reconstruct w(z) in this same approximation, i.e., to use formula (1.6). It is possible in this case to bring all of these calculations to a conclusion and to find the explicit form of w(z). For R(λ), in the place of (1.7) it is convenient to make use of the following modified formula:

$$R_\mu(\lambda)=\frac{-v_0}{2\pi i}\int_{-\infty}^{+\infty}\frac{1-\nu S(x)-iS'(x)}{(x-\mu)(x-\lambda)}\,dx$$

($G^+ \ni \mu$ is a fixed point). With such $R_\mu(\lambda)$ the zero multiples of its denominator are reduced by unity in comparison with (1.7) and coincide with the orders of the poles w(z). Formula (1.9) assumes the form

$$c_m=-\frac{v_0}{2\pi i}\int_{-\infty}^{+\infty}\frac{1-\nu S(x)-iS'(x)}{(x-\mu)(x-\lambda_0)^{m+1}}\,dx. \tag{2.2}$$

Assuming that $\mu = i$, $\lambda_0 = i/2$ and substituting (2.1) into (2.2), we find

$$c_k = \frac{\Gamma}{2\pi} \frac{1}{[(-i)^{k+1}(h+1)(h+1/2)^{k+1}}.$$

Having calculated the elements of the C table C(L, M), we obtain $C(0, 1) = \Gamma/[i2\pi(h + 1)\cdot(h + 1/2)] \neq 0$, $C(\ell, m) = 0$, $\ell = 1, 2, \ldots$, $m = 2, 3, \ldots$, from which, in accordance with (1.12), we conclude that $R(\eta)$ has the form of $R(\eta) = \alpha_0/(\beta_0 + \beta_1\eta)$, $\eta = z + i/2$. The zeros of the denominator in $R(\eta)$ are found by means of (1.11), where L = 0 and M = 1:

$$Q^{[0/1]}(\eta) = \begin{vmatrix} c_0 & c_1 \\ \eta & 1 \end{vmatrix} = 0.$$

Hence it follows that the single zero of the denominator in R(z) is found at the point $z_1 = -ih$, i.e., it coincides with the position of the vortex. Substituting the value of z_1 into (1.15), we have the complex velocity of the flow

$$w(z) = \frac{\Gamma}{2\pi i}\left(\frac{1}{z+ih} + \frac{1}{z-ih}\right) - \frac{\Gamma\nu}{\pi} e^{i\nu z} \int_{-\infty}^{z} \frac{e^{-i\nu t}}{t-ih} dt + v_0.$$

This expression coincides with that derived in [1] through solution of the direct problem.

3. Conclusions. The steady-state problem of the potential streamlining of submerged solids by the proposed method is solved exactly only when the streamlined bodies can be represented by a finite system of point singularities. Let us note that the hypothesis regarding the finiteness of the number of singularities in the complex velocity is used extensively in problems related to the streamlining of solids [1-4, 7]. When the velocity singularities do not reduce to a finite system of poles, we can speak only of an approximate solution. This case is sufficiently complex and goes beyond the scope of the present paper. Let us note that the example considered above is primarily illustrative in nature and of interest only from the standpoint that it allows us to clarify the essential nature of the proposed approach.

4. Stability of the Method. If the profile of the free surface S(x) corresponds to a flow with a complex velocity w(z), the analytical region $D = \{z: \operatorname{Im} z < S(x) + \varepsilon, \varepsilon > 0\}$, with the exception of a finite number of poles $z_i \in G$ of total order P, then (1.11) and (1.14) uniquely reproduce w(z) in G with respect to S(x). Under the initial condition of continuity for the second derivative of the function S(x) we can prove the continuity of the representation $S(x) \to w(z)$ specified in (1.11) and (1.14) in the vicinity of $S^0(x)$ in the following sense. For any $\rho > 0$, $\varepsilon > 0$ we find $\delta > 0$ such that from

$$\sup_{x \in R^1} \{|S(x) - S^0(x)|, |S'(x) - (S^0(x))'|\} < \delta \tag{4.1}$$

it will follow that $|w(z) - w^0(z)| < \varepsilon$ for all $z \in K \subset F$, where K is compact and $F = G^0 \setminus \bigcup_1^N V_i(\rho)$, G^0, w^0 is the region occupied by the fluid, and the complex velocity of the flow, corresponding to the profile $S^0(x)$, N represents the number of poles w in G, $V_i(\rho) = \{z: |z - z_i^0| < \rho\}$.

In order to prove this contention it is sufficient successively to estimate the quantities $|c_n - c_n^0|$, $n = 0, 1, \ldots, 2N$, using (1.9), as well as $|w(z) - w^0(z)|$ on the set F with consideration of the fact that $|z_i - z_i^0| \to 0$ as $\delta \to 0$. Within the scope of linear theory condition (4.1) can be replaced with the weaker

$$\int_{-\infty}^{+\infty} \frac{|S - S^0|^2 + |S' - (S^0)'|^2}{(x^2 + \mu^2)^\alpha} dx < \delta \quad \left(\alpha < \frac{3}{2}\right).$$

The proof is accomplished analogously.

5. Prospects. As was noted in [8], the familiar inconsistency of the surface profiles, the latter obtained theoretically (in the solution of the direct problems) as well as experimentally, is associated with the inadequacy of the streamlining model contained in the theoretical calculations. The proposed method makes it possible to enhance this interesting remark with practical results. Indeed, the problem of localizing the point poles of the velocity allows an exact solution. Consequently, on the basis of the experimentally derived

profiles of the free surface it is possible to ascertain the nature of streamlining for various bodies: under which conditions a given body allows representation by point poles, the determination of their orders and position, as well as under which conditions the "point" representation is violated. For example, we can study the case of the streamlining of a cylinder which is traditionally simulated by a point dipole. The streamlining potential for a solid is not an additive quantity [7], so that in the case in which the cylinder is in motion in the immediate vicinity of the free surface its influence is therefore significant, so that it is possible to assume that the potential of the dipole ceases satisfactorily to describe the streamlining of the cylinder. In view of the stability of the proposed method the vanishingly small errors in the measurement of the input data (the surface profile) introduce vanishingly small errors into the solution, i.e., the substitution of similar problems in the context of the proposed approach is valid.

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THE EVOLUTION OF WEAKLY LINEAR PERTURBATIONS IN A PLUG FORMED OF AN AIR—WATER MIXTURE

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UDC 532.529

Various flow regimes arise in the motion of gas and vapor mixtures combined with liquids (a bubble flow, a plug-type flow, a rodlike flow, etc.), and these various types of flows are distinguished on the basis of their hydrodynamic and gasdynamic characteristics. At the present time, the formation and propagation of pressure waves in a mixture of a liquid with gas bubbles have been studied rather thoroughly, both from the theoretical and experimental standpoints. As regards the plug-type regime of flow in a gas—liquid mixture, existing information [1-4] is insufficient to comprehend the entire pattern involved in the process of wave formation. Initially, the model for the propagation of pressure waves was proposed independently in [3, 4], where it was assumed that the propagation of a wave in such a medium comes about as a result of inertialess compression and expansion of the gas plug and through the transfer of momentum to the liquid plug. It was demonstrated in [3, 5] that the mathematical description of the evolution of the waves is reduced, as in a bubble medium, to an equation of the Korteweg—de Vries type, and here we find also a hypothesis dealing with the possibility of forming pressure waves in such a medium, where the shape and quantitative relationships for the propagation of these waves are identical to those that prevail in a gas—liquid bubble mixture.

Novosibirsk. Translated from Zhurnal Prikladnoi Mekhaniki i Tekhnicheskoi Fiziki, Vol. 30, No. 6, pp. 91-98, November-December, 1989. Original article submitted February 16, 1988; revision submitted August 5, 1988.

 0021-8944/89/3006-0920$12.50